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In this work we consider Randall-Sundrum brane-world type scenarios, in which the spacetime is described by a five-dimensional manifold with matter fields confined in a domain wall or three-brane. We present the results of a systematic analysis, using dynamical systems techniques, of the qualitative behaviour of the Friedmann-Lemaître-Robertson-Walker and the Bianchi I and V cosmological models in these scenarios. We construct the state spaces for these models and discuss how their structure changes with respect to the general-relativistic case, in particular, what new critical points appear and their nature, the occurrence of bifurcations and the dynamics of anisotropy.

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I. INTRODUCTION

String and membrane theories are promising candidates for a unified theory of all forces and particles in Nature. A consistent construction of a quantum string theory is only possible in more than four spacetime dimensions. Then, to make a direct connection of these theories with our familiar non-compact four-dimensional spacetime we are compelled to compactify the extra spatial dimensions to a finite size or, alternatively, find a mechanism to localize matter fields and gravity in a lower dimensional submanifold.

Recently, Randall and Sundrum have shown that for non-factorizable geometries in five dimensions there exists a single massless bound state confined in a domain wall or three-brane [1]. This bound state is the zero mode of the Kaluza-Klein dimensional reduction and corresponds to the four-dimensional graviton. The picture of this scenario is a five-dimensional Anti-de Sitter space (*bulk*) with an embedded three-brane where matter fields are confined and Newtonian gravity is effectively reproduced at large-scale distances. Earlier work on Kaluza-Klein dimensional reduction and matter localization in a four-dimensional manifold of a higher-dimensional non-compact spacetime can be found in [2].

The Randall-Sundrum model was inspired by string theory. In the context of dimensional reduction of eleven-dimensional supergravity, Hořava and Witten showed that the ten-dimensional $E_8 \times E_8$ heterotic string is connected with an eleven-dimensional theory compactified on the orbifold $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$ [3]. Moreover, they concluded that the coupling constants of gauge fields in the ten-dimensional boundary are related with the eleven-dimensional gravitational constant [4]. The picture coming out of this model is that of two separated ten-dimensional manifolds. Gauge fields are confined in these boundary manifolds whereas gravity can propagate in the higher dimensional spacetime. As a consequence, these two separated worlds can only communi-

cate through gravitational interactions. The cosmological implications of the Hořava-Witten theory have already been extensively analyzed [5].

The original motivation for the Randall-Sundrum model was the solution of the hierarchy problem in a slightly different set up [6]. In this case one has two parallel branes with opposite tensions embedded in a five-dimensional spacetime with negative cosmological constant. Actually, the fifth dimension is compactified in the orbifold S^1/\mathbb{Z}_2 and the two branes are located at the singular boundary points. Due to an exponential factor in the metric tensor, the particles living in the negative tension brane acquire effectively a huge physical mass parameter compared to the fundamental scale with a moderate fine tuning of the size of the extra dimension. Unfortunately, as pointed out in [7], the cosmology in this brane is rather unsatisfactory because the energy density of matter present in the brane must be negative, which violates the weak energy condition. The argument is based on the observation made by Binétruy *et al.* [8,9] (see also [10]) that the effective Friedmann equation for the Hubble parameter for a five-dimensional spacetime with energy density localized in a infinitely thin domain wall is modified with respect to the general relativistic case. Other attempts to solve the hierarchy problem in the context of extra dimension have been examined in [11].

Nevertheless, the model with a non-compact fifth dimension and only one brane is consistent with present gravity experiments. In general, scenarios with extra dimensions predict corrections to the Newtonian potential at short distances and important deviations from the standard evolution of the universe at early times. Then, current day cosmological observations, such as the age of the universe or the abundances of light elements, cannot be used to constraint these models. In contrast, the search for deviations of Newton's law is their fundamental observational probe [12]. The fact that Newtonian gravity has been tested quite accurately up to 1 mm

($M_* \sim 10^{-16} \text{ TeV}$) limits the value of the fundamental scale associated with the five-dimensional gravitational coupling constant: $M^{(5)}_P \gg M_P^{2/3} M_*^{1/3} \sim 10^5 \text{ TeV}$ [1,13] (M_P is the Planck mass). Future experiments will further constraint this naive estimate [14,15]

The purpose of the present work is to study the cosmological evolution of these brane-world scenarios. We are going to follow the geometric formulation and generalization of the Randall-Sundrum scenario introduced in [16–18]. The Einstein equations in the bulk can be written in the following form*

$$G_{AB}^{(5)} = -\Lambda_{(5)} g_{AB}^{(5)} + \kappa_{(5)}^2 T_{AB}^{(5)}, \quad (1)$$

with

$$T_{AB}^{(5)} = \delta(\chi) [-\lambda g_{AB} + T_{AB}]. \quad (2)$$

In these expressions $\kappa_{(5)}$ is the five-dimensional gravitational coupling constant; $g_{AB}^{(5)}$, $G_{AB}^{(5)}$ and $\Lambda_{(5)}$ are the metric, Einstein tensor and the cosmological constant of the bulk spacetime, respectively; T_{AB} is the matter energy-momentum tensor; the spacelike hypersurface $x^4 \equiv \chi = 0$ gives the brane world and g_{AB} is its induced metric; finally, λ is the tension of the brane, which must be assumed to be positive in order to recover conventional gravity on the brane. Using the Gauss-Codacci equations relating the four- and five-dimensional spacetimes, equations (1-2) lead to the following modification of the Einstein's equations of General Relativity on the brane [16,17]

$$G_{ab} = -\Lambda g_{ab} + \kappa^2 T_{ab} + \kappa_{(5)}^4 S_{ab} - E_{ab}^{(5)}, \quad (3)$$

where g_{ab} is the four-dimensional metric on the brane and G_{ab} its Einstein tensor. The four-dimensional gravitational constant κ and the cosmological constant Λ are given in terms of the fundamental constants in the bulk by

$$\kappa^2 = \frac{1}{6} \lambda \kappa_{(5)}^4,$$

$$\Lambda = \frac{|\Lambda_{(5)}|}{2} \left[\left(\frac{\lambda}{\lambda_c} \right)^2 - 1 \right],$$

respectively; where λ_c is a critical brane tension[†] given by

*Upper-case Latin letters denote coordinate indices in the bulk spacetime ($A, B, \dots = 0, \dots, 4$) whereas lower-case Latin letters denote coordinate indices in the four-dimensional spacetime where matter is confined ($a, b, \dots = 0, \dots, 3$). We will use physical units in which $c = 1$.

[†]The particular Randall-Sundrum solution corresponds to the case when the tension of the brane λ equals the critical brane tension λ_c (4) and $T_{ab} = E_{ab}^{(5)} = 0$.

$$\lambda_c = 6 \frac{|\Lambda_{(5)}|}{\kappa_{(5)}^2}. \quad (4)$$

S_{ab} are corrections quadratic in the matter variables (due to the form of the Gauss-Codacci equations) and given by

$$S_{ab} = \frac{1}{12} T T_{ab} - \frac{1}{4} T_a^c T_{bc} + \frac{1}{24} g_{ab} [3 T^{cd} T_{cd} - T^2], \quad (5)$$

where $T \equiv T_a^a$. And finally, $E_{ab}^{(5)}$ are corrections coming from the extra dimension, more precisely, $E_{ab}^{(5)}$ are the components of the *electric part* of the Weyl tensor of the bulk, $C_{ABCD}^{(5)}$, with respect to the normal, n_A ($n^A n_A = 1$), to the hypersurface $\chi = 0$ where matter is confined, that is,

$$E_{AB}^{(5)} = C_{ACBD}^{(5)} n^C n^D.$$

Moreover, it is worth to note that the twice contracted second Bianchi identities in the bulk, $\nabla_{(5)}^A G_{AB}^{(5)} = 0$, imply

$$\nabla_a T^a_b|_{\chi=0} = 0, \quad (6)$$

where we have taken $\{\chi, x^a\}$ to be Gaussian normal coordinates (see, e.g., [19]) adapted to the hypersurface $\chi = 0$. Therefore, we can say that the Einstein equations in the bulk (1) imply the conservation of the energy-momentum tensor in the brane world.

In this paper we will deal with generalized Randall-Sundrum scenarios in which the effects of the extra-dimension come from the term quadratic in the energy-momentum tensor, i.e. S_{ab} (5). Thus, we are assuming

$$E_{AB}^{(5)}|_{\chi=0} = 0 \iff E_{ab}^{(5)} = 0. \quad (7)$$

This includes conformally-flat bulks ($C_{ABCD} = 0$), and in particular, the five-dimensional Anti-de Sitter spacetime, the bulk considered in the original Randall-Sundrum scenario. The extension of this work to general bulks will be presented in a future paper [20].

For the scenarios just outlined we have constructed and studied the state space of the Friedmann-Lemaître-Robertson-Walker (FLRW) and the Bianchi I and V cosmological models. Then, we have discussed systematically how the extra dimension changes the dynamics with respect to the general-relativistic case. In particular, we find a new critical point representing the dynamics at very high energies, in the early universe (near the Big-Bang) and also near the Big-Crunch in the case of recollapsing models. We also find new bifurcations in the state space as the equation of state of matter changes (we will assume a perfect-fluid energy-momentum content), which are characterized by the occurrence of an infinite number of non-general-relativistic critical points. Finally, the Bianchi I and V models will provide information regarding the dynamics of anisotropy in the brane-world scenario.

The paper is organized as follows. In section II we will study the dynamics of the FLRW models in the brane-world scenarios, introducing the notation and some tools used in the analysis of dynamical systems (see, e.g., [21,22]). In section III, we will study the dynamics in homogeneous but anisotropic cosmological models. In particular, the dynamics of the orthogonal Bianchi I and V cosmological models, which contain the flat and the negatively curved FLRW models, respectively. We will finish with some concluding remarks in section IV.

II. DYNAMICS OF THE FLRW MODELS IN THE BRANE-WORLD SCENARIO

In this section, we start by assuming that the brane-world is described by a FLRW metric. The FLRW spacetimes are the standard cosmological models. As is well-known [23,24], they are motivated by the so-called *Cosmological Principle* in the sense that they are homogeneous and isotropic cosmological models (they have a six-dimensional group of motions). Then, the line element in the brane-world ($\chi = 0$) will be given by

$$ds^2 = -dt^2 + a^2(t) [dr^2 + \Sigma_k^2(r)(d\theta^2 + \sin^2\theta d\varphi^2)],$$

where

$$\Sigma_k(r) = \begin{cases} \sin r & \text{for } k = 1, \\ r & \text{for } k = 0, \\ \sinh r & \text{for } k = -1, \end{cases}$$

and $a(t)$ is the scale factor.

Here, we will study the dynamics of the FLRW models considering a bulk spacetime satisfying the condition (7), which includes the five-dimensional Anti-de Sitter spacetime. On the other hand, we will assume that the matter content is equivalent to that of a perfect fluid and therefore, the energy-momentum tensor will have the following form

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab},$$

where \mathbf{u} , ρ and p are the unit fluid velocity of matter ($u^a u_a = -1$), the energy density and the pressure of the matter fluid respectively. We will also assume a linear barotropic equation of state for the fluid, that is,

$$p = (\gamma - 1)\rho. \quad (8)$$

The weak energy condition (see, e.g., [23]) imposes the restriction $\rho \geq 0$, and from causality requirements, the speed of sound [$c_s \equiv (dp/d\rho)^{1/2}$] must be less than the speed of light, we have that $\gamma \in [0, 2]$. Then, taking into account the form of the equations (3,5) and (7), it turns out that the fluid velocity \mathbf{u} is aligned with the velocity of the preferred observers in the FLRW spacetimes (excepting in the case $G_{ab} \propto g_{ab}$, where there are no preferred observers), those that observe the matter distribution to

be homogeneous and isotropic. Then, we can write \mathbf{u} as follows

$$\vec{u} = \frac{\partial}{\partial t}, \quad \implies \quad \mathbf{u} = -d\mathbf{t}.$$

Finally, taking into account recent observations [25,26], we will consider only the case of a positive cosmological constant, i.e. $\Lambda \geq 0$. Then, introducing the Hubble function $H(t)$

$$H(t) \equiv \frac{1}{a} \frac{da}{dt} \equiv \frac{\dot{a}}{a},$$

the dynamics of the FLRW models imposed by the modified Einstein field equations (3) and the energy-momentum conservation equation (6) is governed by the following set of ordinary differential equations

$$\dot{H} = -H^2 - \frac{3\gamma - 2}{6} \kappa^2 \rho \left[1 + \frac{3\gamma - 1}{3\gamma - 2} \frac{\rho}{\lambda} \right] + \frac{1}{3} \Lambda, \quad (9)$$

$$\dot{\rho} = -3\gamma H \rho, \quad (10)$$

$$H^2 = \frac{1}{3} \kappa^2 \rho \left(1 + \frac{\rho}{2\lambda} \right) - \frac{1}{6} {}^3R + \frac{1}{3} \Lambda, \quad (11)$$

where 3R denotes the scalar curvature of the hypersurfaces orthogonal to the fluid velocity, the $\{t = \text{constant}\}$ hypersurfaces, which is given by ${}^3R = 6ka^{-2}(t)$. Equation (9) is the modified Raychaudhuri equation, equation (10) comes from the energy-momentum tensor conservation equation, and finally, equation (11) is the modified Friedmann equation. As is well-known, (9) is a consequence of (10) and (11), and the dynamics is completely described by the functions (H, ρ) and the parameters k , κ , γ , Λ and λ .

In order to study the dynamics of these models we will closely follow the analysis carried out by Goliath & Ellis [27] for general relativistic FLRW models with a cosmological constant. To that end, and in order to get compactified state spaces, it is convenient to consider two differentiated cases: (i) ${}^3R \leq 0$ ($k = 0$ or $k = -1$) and (ii) ${}^3R > 0$ ($k = 1$).

In the case (i), let us introduce the following set of dimensionless variables

$$\Omega_\rho \equiv \frac{\kappa^2 \rho}{3H^2}, \quad \Omega_k \equiv -\frac{{}^3R}{6H^2} = -\frac{k}{\dot{a}^2}, \quad (12)$$

$$\Omega_\Lambda \equiv \frac{\Lambda}{3H^2}, \quad \Omega_\lambda \equiv \frac{1}{6\lambda} \frac{\kappa^2 \rho^2}{H^2}, \quad (13)$$

where Ω_ρ is the ordinary density parameter and Ω_k , Ω_Λ and Ω_λ are the fractional contributions of the curvature, cosmological constant and brane tension, respectively, to the universe expansion (11). Therefore, all of them have a clear physical meaning. As we can see, they are non-negative and singular when $H = 0$. Furthermore, the

Friedmann equation (11), which now takes the following simple form

$$\Omega_\rho + \Omega_k + \Omega_\Lambda + \Omega_\lambda = 1. \quad (14)$$

implies that they must belong to the interval $[0, 1]$ and hence, the state space with coordinates $\mathbf{\Omega} = (\Omega_\rho, \Omega_k, \Omega_\Lambda, \Omega_\lambda)$ is compact.

In order to find the dynamical equations for these variables we will introduce the following dimensionless time derivative

$$' \equiv \frac{1}{|H|} \frac{d}{dt}, \quad (15)$$

where $|H|$ is the absolute value of H . Then, we have

$$H' = -\epsilon(1 + q)H, \quad (16)$$

where ϵ is the sign of H [$\epsilon \equiv \text{sgn}(H)$]. As is clear, for $\epsilon = 1$ the model will be in expansion, and for $\epsilon = -1$ it will be in contraction. Moreover, q is the deceleration parameter, which is defined by

$$q \equiv -\frac{1}{H^2} \frac{\ddot{a}}{a} = \frac{3\gamma - 2}{2} \Omega_\rho - \Omega_\Lambda + (3\gamma - 1) \Omega_\lambda.$$

Then, the dynamical system for our dimensionless variables (12,13) can be written in the following form

$$\Omega'_\rho = \epsilon[2(1 + q) - 3\gamma] \Omega_\rho, \quad (17)$$

$$\Omega'_k = 2\epsilon q \Omega_k, \quad (18)$$

$$\Omega'_\Lambda = 2\epsilon(1 + q) \Omega_\Lambda, \quad (19)$$

$$\Omega'_\lambda = 2\epsilon(1 + q - 3\gamma) \Omega_\lambda. \quad (20)$$

It is important to note that equation (16) is not coupled to the system of equations (17-20), and therefore we can ignore it for the dynamical analysis. To begin with, we have to find the critical points of this dynamical system, which can be written in vector form as follows

$$\mathbf{\Omega}' = \mathbf{f}(\mathbf{\Omega}),$$

where \mathbf{f} can be extracted from (17-20). The critical points, $\mathbf{\Omega}^*$, which are the points at which the system will stay if initially it was there (see, e.g. [22]), are given by the condition

$$\mathbf{f}(\mathbf{\Omega}^*) = \mathbf{0}.$$

Their dynamical character is determined by the eigenvalues of the matrix

$$\left. \frac{\partial \mathbf{f}}{\partial \mathbf{\Omega}} \right|_{\mathbf{\Omega}=\mathbf{\Omega}^*}.$$

If the real part of the eigenvalues of a critical point is not zero, the point is said to be *hyperbolic*. In this case, the dynamical character of the critical point is determined by the sign of the real part of the eigenvalues: If all

of them are positive, the point is said to be a *repeller*, because arbitrarily small deviations from this point will move the system away from this state. If all of them are negative the point is called an *attractor* because if we move the system slightly from this point in an arbitrary way, it will return to it. Otherwise, we say the critical point is a *saddle* point. The dynamical system (17-20) has four hyperbolic critical points corresponding to: the flat FLRW models (F), $k = \Lambda = \lambda^{-1} = 0$ and $a(t) = t^{2/(3\gamma)}$; the Milne universe (M), $\rho = \Lambda = 0$, $k = -1$ and $a(t) = t$; the de Sitter model (dS), $k = \rho = 0$ and $a(t) = \exp(\sqrt{\Lambda/3}t)$; and a non-general-relativistic model (m) first discussed by Binétruy, Deffayet and Langlois [8] in a brane-world scenario without brane tension (see [9,10] for more details). Their coordinates in the state space, i.e., $\mathbf{\Omega} = (\Omega_\rho, \Omega_k, \Omega_\Lambda, \Omega_\lambda)$, and their eigenvalues are given in the following table [28]

Model	Coordinates	Eigenvalues
F $_\epsilon$	(1, 0, 0, 0)	$\epsilon(3\gamma - 2, 3\gamma - 2, 3\gamma, -3\gamma)$
M $_\epsilon$	(0, 1, 0, 0)	$\epsilon(-(3\gamma - 2), 0, 2, -2(3\gamma - 1))$
dS $_\epsilon$	(0, 0, 1, 0)	$-\epsilon(3\gamma, 2, 2, 6\gamma)$
m $_\epsilon$	(0, 0, 0, 1)	$\epsilon(3\gamma, 2(3\gamma - 1), 6\gamma, 2(3\gamma - 1))$

The dynamical character of these points is given in a table below.

Now, let us consider the situation of the case (ii), in which 3R is positive. As we have already mentioned, in this case the state space defined by the variables $\mathbf{\Omega} = (\Omega_\rho, \Omega_k, \Omega_\Lambda, \Omega_\lambda)$ is no longer compact (because now $\Omega_k < 0$). However, we can introduce another set of variables, analogous to the ones introduced previously (12,13), describing a compact state space. Firstly, instead of using the Hubble function H we will use the following quantity

$$D \equiv \sqrt{H^2 + \frac{1}{6}{}^3R}, \quad (21)$$

and from it, let us define the following dimensionless variables

$$Q \equiv \frac{H}{D}, \quad \tilde{\Omega}_\rho \equiv \frac{\kappa^2 \rho}{3D^2}, \quad (22)$$

$$\tilde{\Omega}_\Lambda \equiv \frac{\Lambda}{3D^2}, \quad \tilde{\Omega}_\lambda \equiv \frac{1}{6\lambda} \frac{\kappa^2 \rho^2}{D^2}.$$

From these definitions we see that now the case $H = 0$ is included. Moreover, the Friedmann equation takes the following form

$$\tilde{\Omega}_\rho + \tilde{\Omega}_\Lambda + \tilde{\Omega}_\lambda = 1, \quad (23)$$

which, together with the fact that $-1 \leq Q \leq 1$ [see Equation (22)], implies that the state space defined by the new variables is indeed compact. Using the following new time derivative

$$' \equiv \frac{1}{D} \frac{d}{dt}, \quad (24)$$

the system of evolution equations for the variables D , Q , $\tilde{\Omega}_\rho$, $\tilde{\Omega}_\Lambda$, and $\tilde{\Omega}_\lambda$ is given by

$$\begin{aligned} D' &= -(1 + qQ^2)QD, \\ Q' &= -qQ^2(1 - Q^2), \end{aligned} \quad (25)$$

$$\tilde{\Omega}'_\rho = [2(1 + qQ^2) - 3\gamma] Q\tilde{\Omega}_\rho, \quad (26)$$

$$\tilde{\Omega}'_\Lambda = 2(1 + qQ^2)Q\tilde{\Omega}_\Lambda, \quad (27)$$

$$\tilde{\Omega}'_\lambda = 2[1 + qQ^2 - 3\gamma] Q\tilde{\Omega}_\lambda, \quad (28)$$

where the deceleration parameter is now given by

$$1 + qQ^2 = \frac{3\gamma}{2}(\tilde{\Omega}_\rho + 2\tilde{\Omega}_\Lambda),$$

The evolution equation for D is not coupled to the rest, so we will not consider it for the dynamical study. Thus, we study the dynamical system for the variables $\tilde{\Omega} \equiv (Q, \tilde{\Omega}_\rho, \tilde{\Omega}_\Lambda, \tilde{\Omega}_\lambda)$, determined by the equations (25-28). The complete set of critical points, their coordinates in the state space, i.e. $\tilde{\Omega}^*$, and their corresponding eigenvalues are given in the following table [28]

Model	Coordinates	Eigenvalues
F_ϵ	$(\epsilon, 1, 0, 0)$	$\epsilon(3\gamma - 2, 3\gamma, 3\gamma, -3\gamma)$
dS_ϵ	$(\epsilon, 0, 1, 0)$	$-\epsilon(2, 3\gamma, 0, 6\gamma)$
E	$(0, \tilde{\Omega}_\rho^*, \tilde{\Omega}_\Lambda^*, \tilde{\Omega}_\lambda^*)$	$(0, \sqrt{\phi}, 0, -\sqrt{\phi})$
m_ϵ	$(\epsilon, 0, 0, 1)$	$2\epsilon(3\gamma - 1, \frac{3\gamma}{2}, 3\gamma, 3\gamma)$

Where $\tilde{\Omega}_\rho^*$, $\tilde{\Omega}_\Lambda^*$ and $\tilde{\Omega}_\lambda^*$ are constants satisfying (23) and the relations

$$\tilde{\Omega}_\rho^* = 2 \left(\frac{1}{3\gamma} - \tilde{\Omega}_\lambda^* \right), \quad \tilde{\Omega}_\Lambda^* = 1 - \frac{2}{3\gamma} + \tilde{\Omega}_\lambda^*. \quad (29)$$

Here, E represents a set of infinite saddle points whose line element is that of the Einstein universe ($k = 1$ and $H = 0$). The eigenvalues of these points are determined by ϕ , which in terms of $\tilde{\Omega}^*$ and γ is given by

$$\phi = \frac{3\gamma}{2} \left[(3\gamma - 2)\tilde{\Omega}_\rho^* + 4(3\gamma - 1)\tilde{\Omega}_\lambda^* \right].$$

One can check, using (23) and (29), that ϕ is always positive. The dynamical character of all the equilibrium points is given in the table below. As we can see from the previous tables, it depends on the equation of state (on the parameter γ) and on the expanding ($H > 0 \Leftrightarrow \epsilon = 1$) or contracting character ($H < 0 \Leftrightarrow \epsilon = -1$) of the point:

Model	Dynamical character		
	$0 < \gamma < \frac{1}{3}$	$\gamma = \frac{1}{3}$	$\gamma > \frac{1}{3}$
F_\pm	saddle	saddle	saddle
M_+	repeller	repeller	saddle
M_-	attractor	attractor	saddle
dS_+	attractor	attractor	attractor
dS_-	repeller	repeller	repeller
E	—	saddle	saddle
m_+	saddle	repeller	repeller
m_-	saddle	attractor	attractor

At this point, we can observe some differences with the general relativistic case [27]. First, the Einstein Universe (E) appears to be a critical point for $\gamma \geq \frac{1}{3}$, in contrast with general relativity, where it appears for $\gamma \geq \frac{2}{3}$. On the other hand, as we will discuss in detail later, the dynamical character of some of the points changes with respect to general relativity. For instance, in the brane-world scenario the expanding and contracting flat FLRW models (F_+ and F_- respectively) are no longer repeller and attractor, respectively, for $\gamma > \frac{2}{3}$. They are now saddle points.

Another important difference is that now we have additional critical points, namely, m_+ and m_- . Let us analyze in detail the dynamics represented by these models. First of all, we have to point out that their characterization presents an extra difficulty with respect to the other models. Their coordinates in the state space are $\tilde{\Omega}^* = (0, 0, 0, 1)$ and $\tilde{\Omega}^* = (\epsilon, 0, 0, 1)$, i.e. the contributions of the ordinary matter term (Ω_ρ), the spatial curvature (Ω_k) and the cosmological constant (Ω_Λ) are negligible. Therefore, we have at the same time $\kappa^2 \rho H^{-2} \rightarrow 0$ and $(6\lambda)^{-1} \kappa^2 \rho^2 H^{-2} \rightarrow 1$, hence their characterization must involve a limiting process. In order to understand the dynamics let us consider the simplified situation $\Lambda = {}^3R = 0$, in which the Friedmann equation (11) can be solved to give

$$a(t) = (t - t_{BB})^{\frac{1}{3\gamma}} (t + t_{BB})^{\frac{1}{3\gamma}}, \quad (30)$$

where the constant t_{BB} is the Big-Bang time

$$t_{BB} \equiv \sqrt{\frac{2}{3\gamma^2 \kappa^2 \lambda}} = \frac{1}{3\gamma} \left(\frac{\kappa_{(5)}}{\kappa} \right)^2.$$

In the state-space diagrams shown in the Figures II-II below, this situation corresponds to models in the line joining m_+ and F_+ . From (30), we deduce that for late times, $t \gg t_{BB}$, the scale factor behaves as $a(t) \sim t^{\frac{2}{3\gamma}}$, and therefore the solution approaches the flat FLRW model (F_+), hence we have a general relativistic behaviour. However, the new interesting behaviour appears when we approach the initial singularity ($t \rightarrow t_{BB}$) or, in other words, at very high energies ($\rho \gg \lambda$), where we have $a(t) \sim (t - t_{BB})^{\frac{1}{3\gamma}}$. From the point of view of Einstein's equations (3), in such a situation the term involving the four-dimensional constant, κ , becomes negligible with respect to the term involving the five-dimensional one, $\kappa_{(5)}$. We recover general relativity in the limit $t_{BB} \rightarrow 0$, which is the opposite situation. From this discussion we realize that the limiting process leading to the critical points m_\pm is

$$\kappa \rightarrow 0 \quad \left(\frac{\kappa^2}{6\lambda} \rightarrow \frac{1}{\kappa_{(5)}^4} \right). \quad (31)$$

Then, we find that the points m_\pm are models whose scale factor is given by

$$a(t) = t^{\frac{1}{3\gamma}}. \quad (32)$$

This is the Binétry-Deffayet-Langlois (BDL) solution [8] (see also [9,10]) As we have already mentioned, these models describe the dynamics near the singularities. That is, the early universe behaviour, near the initial Big-Bang singularity and also, for recollapsing models (for which we must have ${}^3R > 0$), the dynamical behaviour when we approach the Big-Crunch singularity. In both cases the dynamics changes with respect to general relativity.

With the information we have obtained about the critical points of the dynamical systems for Ω and $\tilde{\Omega}$, we can apply the well-known techniques used in dynamical systems [22] to obtain the structure of the state space, which provides, in a visual way, the complete information on the evolution of our system (a perfect-fluid FLRW model in the brane-world scenario) once the initial conditions are given. In the same way as the dynamical character of the critical points depend on the equation of state, or equivalently, on the parameter γ , so will do the state space. In fact, we have found that there are values of γ for which *bifurcations*, that is, topological changes in the state space (see [22] for details), appear. Specifically, these values are $\gamma_B = 0, \frac{1}{3}, \frac{2}{3}$ (in general relativity we only have bifurcations at $\gamma = 0, \frac{2}{3}$). As we will see in the discussion of each particular case, for $\gamma = \frac{1}{3}$ and $\gamma = \frac{2}{3}$, we have lines with an infinite number of critical points, for which we get one vanishing eigenvalue, as is expected in those cases [22].

Let us begin with the $\gamma = 0$ case. We have not drawn the state space because it is quite simple. Equation (10) implies that the energy density is constant. Then, we can solve the Friedmann equation (11) and we find that $a(t)$ is given by

$$a(t) = \begin{cases} \sqrt{\frac{3}{\tilde{\Lambda}}} \cosh \left[\epsilon \sqrt{\frac{\tilde{\Lambda}}{3}} (t - t_i) \right] & \text{for } k = 1, \\ e^{\epsilon \sqrt{\frac{\tilde{\Lambda}}{3}} (t - t_i)} & \text{for } k = 0, \\ \sqrt{\frac{3}{\tilde{\Lambda}}} \sinh \left[\epsilon \sqrt{\frac{\tilde{\Lambda}}{3}} (t - t_i) \right] & \text{for } k = -1, \end{cases} \quad (33)$$

where t_i is a constant, $\epsilon = \text{sgn}(H)$, and $\tilde{\Lambda}$ is a modified cosmological constant given by

$$\tilde{\Lambda} = \Lambda + \kappa^2 \rho \left(1 + \frac{\rho}{2\lambda} \right). \quad (34)$$

For $\tilde{\Lambda} \neq 0$, all the models belong to the de Sitter class, whereas in the limit $\tilde{\Lambda} \rightarrow 0$ ($\Lambda = \rho = 0$) we find the Minkowski ($k = 0$) and Milne ($k = -1$) spacetimes. The dynamics (of expanding models, $\epsilon = 1$) is reduced to the fact that the model $k = 0$ is the future attractor, and the Milne universe is a repeller.

For the other cases ($\gamma \neq 0$), the whole state space is constructed by matching the state space corresponding to the dynamical systems (17-20) and (25-28). It consists of three pieces, the diagram shown in Figure II(b)

on the right, which corresponds to the case $\epsilon = 1$ in (17-20), the diagram in Figure II(a) in the middle, and on the left the diagram corresponding to the case $\epsilon = -1$ in (17-20), which has not been included here because it can be obtained from the Figure II(b) just by reversing the direction of the arrows and replacing the subscript “+” by “-”. In order to follow the evolution, we have specified the quantities represented in the different axes. Notice that the state space is compact, with the boundaries given by the planes $\Omega_\Lambda = \tilde{\Omega}_\Lambda = 0$, $\Omega_\lambda = \tilde{\Omega}_\lambda = 0$ and the vacuum models $\Omega_\rho = \tilde{\Omega}_\rho = 0$.

We have drawn only the trajectories on the planes, but the trajectory of any point in the state space outside these planes can be deduced qualitatively following the behaviour shown in them. As is obvious, the general relativistic state space corresponds to the plane $\Omega_\lambda = 0$, which is an *invariant submanifold*[‡] of the state space. Therefore, the aim of this work is to study what happens when we take initial conditions outside of this plane. The other invariant submanifolds are: the vacuum boundary $\Omega_\rho = 0$, the flat geometry submanifold $\Omega_k = 0$, and the $\Omega_\Lambda = 0$ submanifold.

Keeping this preamble in mind, let us analyze the different cases according to γ . For $\gamma \in (0, \frac{1}{3})$ and ${}^3R \leq 0$, Milne is a repeller, as in the general-relativistic case, and the expanding de Sitter model is the future attractor for all the initial conditions excepting the plane $\Omega_\Lambda = 0$, for which the attractor is the flat FLRW model. For ${}^3R > 0$, dS₊ plays the same role. In the plane $\Omega_\Lambda = 0$, collapsing FLRW models evolve towards the expanding flat FLRW model (F₊), with the effect of the extra dimension being maximum when $H = 0$ ($\Leftrightarrow Q = 0$). In conclusion, the dynamics in this case is essentially the same as in general relativity.

The next case, $\gamma = \frac{1}{3}$, constitutes a bifurcation. The topology of the state space changes [see Figures II(a) and II(b)] due to the fact that we have now a line of vacuum critical points. This line extends to the three parts of the whole state space. In the ${}^3R \geq 0$ sector, Figure II(a), all these critical points, excepting the points m_\pm and E, are not included in the previous tables. Their coordinates are $\tilde{\Omega}^* = (Q^*, 0, 0, 1)$, $|Q^*| < 1$, and hence they do not appear in general relativity. In order to see to what particular models they corresponds we need to consider the limit (31) since they have $\tilde{\Omega}_\lambda^* = 1$. Then, solving the Friedmann equation (11), we find they are positively curved FLRW models with dynamics described by $a(t) \sim t$. The particular case $Q^* = 0$ corresponds to the Einstein universe. In the ${}^3R \leq 0$ sector, Figure II(b), the critical points, excepting points m_\pm and M_\pm , are also not in the tables above and they are non-general-relativistic in nature. Their coordinates are $\Omega^* = (0, \Omega_k^*, 0, \Omega_\lambda^*)$ with

[‡]State space trajectories starting in an invariant submanifold will never leave it.

[height=3in,width=3in, bblx=55, bblly=200, bburx=540, bbury=680]center_0 - 0.3_gamma_wl.ps[height = 3in,width = 3in,bblx = 55,bblly = 200,bburx = 540,bbury = 680]right_0 - 0.3_gamma_wl.ps

FIG. 1. State space for the FLRW models with $\gamma \in (0, \frac{1}{3})$ and (a) non-negative spatial curvature, ${}^3R \geq 0$ (on the left) and (b) non-positive spatial curvature, ${}^3R \leq 0$ (on the right). Replacing Ω_k by Ω_σ and M_ϵ by K_ϵ the drawing on the right is also the state space for Bianchi I models with $\gamma \in (0, 1)$. The planes $\tilde{\Omega}_\Lambda - Q$ and $\Omega_\Lambda - \Omega_k$ correspond to the state space of general relativity. The critical points F_ϵ , M_ϵ , dS_ϵ , E , m_ϵ and K_ϵ describe the flat FLRW model, the Milne universe, the de Sitter model, the Einstein universe, the non-general-relativistic BDL models and the Kasner spacetimes respectively. ϵ is the sign of the Hubble function, differentiating between expanding and collapsing models. The planes joining the points dS_ϵ , M_ϵ and m_ϵ represent vacuum solutions ($\Omega_\rho = \tilde{\Omega}_\rho = 0$). Only trajectories on the invariant planes, which outline the whole dynamics, are drawn ($\Omega_k = 0$, $\Omega_\Lambda = \tilde{\Omega}_\Lambda = 0$, and $\Omega_\lambda = \tilde{\Omega}_\lambda = 0$).

$\Omega_k^* + \Omega_\lambda^* = 1$. Then, using the limiting procedure (31), we get the same time dependence: $a(t) \sim t$. These points are also non-general-relativistic.

The next step is to study the state space in the interval $\gamma \in (\frac{1}{3}, \frac{2}{3})$, which is now described by the diagrams shown in Figure II. We can find some changes with respect to the situation in the previous cases. First, we have an infinite number of critical points corresponding to the Einstein universe, which are arranged in a line determined by equations (29). In the ${}^3R \geq 0$ sector we can see that dS_+ and m_- are attractors of the evolution. Then, this sector of the state space is divided into two regions. The first one consists of those points which will evolve to the de Sitter model (dS_+), which corresponds to the whole ${}^3R \geq 0$ sector in the case $\gamma \in (0, \frac{1}{3})$. The second region is determined by the points which evolve towards the BDL model (m_-) which does not contain any general relativistic point. These trajectories correspond to models collapsing in the future, that is, evolving towards a Big Crunch singularity, where the dynamics is given by (32). It is worth noting that in general relativity recollapsing models only occur for $\gamma > \frac{2}{3}$. These two regions are separated by the surface generated by the trajectories that start from or arrive to the set of critical points representing the Einstein universe (E), which are saddle points. In the ${}^3R \leq 0$ sector the situation is simpler. For a vanishing cosmological constant ($\Omega_\Lambda = 0$) the future attractor are the flat FLRW models (F_+) whereas in the case with a cosmological constant it is the de Sitter model (dS_+).

In $\gamma = \frac{2}{3}$ we have another bifurcation motivated by the appearance of two lines of infinite critical points which join at the general relativistic Einstein universe, given by $\Omega^* = (0, 1, 0, 0)$ (see Figure II). One of the lines is composed by Einstein universe points whose state space coordinates satisfy (29). The other line corresponds to general relativistic models (which were not shown in [27]), and it occupies the three regions. Their points are characterized by $\Lambda = \lambda^{-1} = 0$, and their scale factor grows linearly with time ($H \neq 0$), $a(t) = Ct$. They are perfect-fluid models with equation of state $\rho + 3p = 0$ and energy density given by

$$\kappa^2 \rho = \frac{3(C^2 + k)}{C^2 t^2}.$$

The case $k = -1$ (${}^3R < 0$) and $C = 1$ corresponds to the Milne universe.

The last situation corresponds to the case $\gamma > \frac{2}{3}$, described by the state space drawn in Figure II. The situation in the ${}^3R \geq 0$ sector is now very similar to that showed in the $\gamma \in (\frac{1}{3}, \frac{2}{3})$ case, where two regions appeared according to whether the points evolve to the BDL or to the de Sitter model. The region of points evolving to the BDL model is now bigger. With regard to the ${}^3R \leq 0$ sector, the situation has now changed: for the models without cosmological constant the attractor is the Milne universe (M_+), whereas the flat FLRW models (F_+) are saddle points. For a non-vanishing cosmological constant, the de Sitter model (dS_+) is again the attractor.

III. DYNAMICS OF BIANCHI MODELS IN THE BRANE-WORLD SCENARIO

In this section we will study the dynamics of some homogeneous but anisotropic cosmological models (Bianchi models) in the brane-world scenario. In particular, we will consider the perfect-fluid Bianchi I and V homogeneous cosmological models in which the fluid velocity is non-tilted, which means that the hypersurfaces of homogeneity are orthogonal to the fluid flow. Moreover, we will also consider a linear equation of state (8) for the perfect fluid. We have considered these two particular classes of Bianchi models for simplicity and because they contain the flat and negatively curved FLRW models.

It is well-known that Bianchi models can be described by systems of ordinary differential equations, being the fluid proper time t the only independent variable that appears. The form of the system of ordinary differential equations depends on the parametrization of the models, i.e., on the variables we use to describe them. Here, we will start using the point of view adopted by Ellis and MacCallum [29], where they use an orthonormal tetrad, $\{\mathbf{u}, \mathbf{e}_\alpha\}$ ($\alpha = 1, \dots, 3$), adapted to the fluid velocity

$$\mathbf{u} \cdot \mathbf{u} = -1, \quad \mathbf{u} \cdot \mathbf{e}_\alpha = 0, \quad \mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \delta_{\alpha\beta}.$$

Then, the dynamics can be described in terms of the following variables: (i) The *spatial commutation functions*,

[height=3in,width=3in, bblx=55, bblly=200, bburx=540, bbury=680]center0.3 γ amwal.ps[height = 3in,width = 3in, bblx = 55, bblly = 200, bburx = 540, bbury = 680]right0.3 γ amwal.ps

FIG. 2. State space for the FLRW models with $\gamma = \frac{1}{3}$ (a bifurcation) and (a) non-negative spatial curvature, ${}^3R \geq 0$ (on the left) and (b) non-positive spatial curvature, ${}^3R \leq 0$ (on the right). The drawing on the right is also the state space for Bianchi models with $\gamma = 1$. See the caption of Figure II for more details.

[height=3in,width=3in, bblx=55, bblly=200, bburx=540, bbury=680]center0.3 - 0.6 γ amwal.ps[height = 3in,width = 3in, bblx = 55, bblly = 200, bburx = 540, bbury = 680]right0.3 - 0.6 γ amwal.ps

FIG. 3. State space for the FLRW models with $\gamma \in (\frac{1}{3}, \frac{2}{3})$ and (a) non-negative spatial curvature, ${}^3R \geq 0$ (on the left) and (b) non-positive spatial curvature, ${}^3R \leq 0$ (on the right). The drawing on the right is also the state space for Bianchi models with $\gamma \in (1, 2)$. See the caption of Figure II for more details.

$\gamma^\alpha_{\beta\delta}$, defined by the commutation relations between the spatial basis vectors, $[\vec{e}_\beta, \vec{e}_\delta] = \gamma^\alpha_{\beta\delta} \vec{e}_\alpha$ ($\gamma^\alpha_{[\beta\delta]} = \gamma^\alpha_{\beta\delta}$). Here, we will use the equivalent variables

$$a_\alpha \equiv \frac{1}{2} \gamma^\beta_{\alpha\beta}, \quad n^{\alpha\beta} \equiv \frac{1}{2} \varepsilon^{\delta\kappa(\alpha} \gamma^{\beta)}_{\delta\kappa}.$$

introduced by Schücking, Kundt and Behr (see [29] and references therein) to decompose $\gamma^\alpha_{\beta\delta}$ as follows: $\gamma^\alpha_{\beta\delta} = 2a_{[\beta} \delta^\alpha_{\delta]} + \varepsilon_{\beta\delta\kappa} n^{\alpha\kappa}$. (ii) The *kinematical quantities*. The Hubble function H ($\equiv \nabla_a u^a / 3$) and the components of the shear tensor σ_{ab}

$$\sigma_{ab} \equiv h_a^c h_b^d \nabla_{(c} u_{d)} - H h_{ab},$$

where $h_{ab} \equiv g_{ab} + u_a u_b$ is the orthogonal projector to the fluid velocity \mathbf{u} . (iii) The *matter variables*. In our case only the energy density ρ and the isotropic pressure p , related by an equation of state (8).

In the case of Bianchi models the generalized Friedmann equation reads as follows

$$H^2 = \frac{1}{3} \kappa^2 \rho \left(1 + \frac{\rho}{2\lambda} \right) - \frac{1}{6} {}^3R + \frac{1}{3} \sigma^2 + \frac{1}{3} \Lambda, \quad (35)$$

where $2\sigma^2 \equiv \sigma^{ab} \sigma_{ab}$ and the spatial scalar curvature has the following expression in terms of the spatial commutation functions

$${}^3R = -6a^\alpha a_\alpha - n^{\alpha\beta} n_{\alpha\beta} + \frac{1}{2} (n^\alpha_\alpha)^2. \quad (36)$$

On the other hand, from the Einstein equations (3) we have a constraint on our variables

$$3\sigma_{\alpha\beta} a^\beta - \varepsilon_{\alpha\delta\kappa} n^{\kappa\tau} \sigma^\delta_\tau = 0. \quad (37)$$

To find systems of equations for Bianchi models similar to those described in the FLRW case, we need evolution equations for the new variables: a_α , $n^{\alpha\beta}$ and $\sigma_{\alpha\beta}$. However, it is better to consider them for each particular Bianchi case.

A. Bianchi I perfect-fluid cosmologies

The Bianchi I models are homogeneous and anisotropic cosmological models containing the flat FLRW space-times. We can specialize the triad $\{\mathbf{e}_\alpha\}$ in such a way

that the unit vector fields \mathbf{e}_α are Fermi-Walker propagated along \mathbf{u} and at the same time their commutation functions vanish [29]

$$\gamma^\delta_{\alpha\beta} = 0 \Leftrightarrow a_\alpha = n^{\alpha\beta} = 0. \quad (38)$$

Then, in this case the constraint (37) is identically satisfied. Moreover, in these models the spatial curvature, the curvature of the hypersurfaces orthogonal to the fluid velocity, vanishes, that is,

$${}^3R_{ab} = 0.$$

In particular ${}^3R = 0$, which is a consequence of (36) and (38). Then, in this case the Friedmann equation (35) takes the following form

$$\Omega_\rho + \Omega_\Lambda + \Omega_\sigma + \Omega_\lambda = 1, \quad (39)$$

where Ω_ρ , Ω_Λ and Ω_λ are defined as in the FLRW case [see equations (12,13)], and where we have introduced the following dimensionless quantity associated with the shear

$$\Omega_\sigma \equiv \frac{\sigma^2}{3H^2} = \frac{\sigma^{ab} \sigma_{ab}}{6H^2}. \quad (40)$$

We can construct a state space for the Bianchi I cosmological models by taking the variables $\mathbf{\Omega} = (\Omega_\rho, \Omega_\Lambda, \Omega_\sigma, \Omega_\lambda)$. Then, taking into account that all these quantities are positive by definition, the Friedmann equation (39) implies that we have got a compact state space in which these variable are restricted to the interval $[0, 1]$. Using the time derivative defined in (15) and using the evolution equation for σ^2 [29]

$$(\sigma^2)' = -6H\sigma^2, \quad (41)$$

the system of dynamical equations is given by

$$\Omega'_\rho = \epsilon[2(1+q) - 3\gamma]\Omega_\rho, \quad (42)$$

$$\Omega'_\Lambda = 2\epsilon(1+q)\Omega_\Lambda, \quad (43)$$

$$\Omega'_\sigma = 2\epsilon(q-2)\Omega_\sigma, \quad (44)$$

$$\Omega'_\lambda = 2\epsilon[1+q-3\gamma]\Omega_\lambda, \quad (45)$$

[height=3in,width=3in, bblx=55, bblly=200, bburx=540, bbury=680]center0.6 γ amw.ps[height = 3in,width = 3in,bblx = 55,bblly = 200,bburx = 540,bbury = 680]right0.6 γ amw.ps

FIG. 4. State space for the FLRW models with $\gamma = \frac{2}{3}$ (a bifurcation) and (a) non-negative spatial curvature, ${}^3R \geq 0$ (on the left) and (b) non-positive spatial curvature, ${}^3R \leq 0$ (on the right). The drawing on the right is also the state space for Bianchi models with $\gamma = 2$. See the caption of Figure II for more details.

[height=3in,width=3in, bblx=55, bblly=200, bburx=540, bbury=680]center0.6 - 2 γ amw.ps[height = 3in,width = 3in,bblx = 55,bblly = 200,bburx = 540,bbury = 680]right0.6 - 2 γ amw.ps

FIG. 5. State space for the FLRW models with $\gamma > \frac{2}{3}$ and (a) non-negative spatial curvature, ${}^3R \geq 0$ (on the left) and (b) non-positive spatial curvature, ${}^3R \leq 0$ (on the right). See the caption of Figure II for more details.

and the equation (16), which again is uncoupled to the rest of equations. Now, the expression for the deceleration parameter q in terms of the variables Ω is given by

$$q = \frac{3\gamma - 2}{2}\Omega_\rho - \Omega_\Lambda + 2\Omega_\sigma + (3\gamma - 1)\Omega_\lambda. \quad (46)$$

The critical points of the dynamical system (42-45) having a hyperbolic character, together with their state space coordinates $\Omega^* = (\Omega_\rho^*, \Omega_\Lambda^*, \Omega_\sigma^*, \Omega_\lambda^*)$ and their eigenvalues are given in the following table:

Model	Coordinates	Eigenvalues
F_ϵ	(1, 0, 0, 0)	$\epsilon(3\gamma - 2, 3\gamma, 3(\gamma - 2), -3\gamma)$
dS_ϵ	(0, 1, 0, 0)	$-\epsilon(3\gamma, 2, 6, 6\gamma)$
K_ϵ	(0, 0, 1, 0)	$\epsilon(-3(\gamma - 2), 6, 4, -6(\gamma - 1))$
m_ϵ	(0, 0, 0, 1)	$\epsilon(3\gamma, 6\gamma, 6(\gamma - 1), 2(3\gamma - 1))$

where K denotes the Kasner vacuum spacetimes, whose line element can be written as follows

$$ds^2 = -dt^2 + \sum_{\alpha=1}^3 t^{2p_\alpha} (dx^\alpha)^2, \quad (47)$$

where p_α are constants satisfying

$$\sum_{\alpha=1}^3 p_\alpha = \sum_{\alpha=1}^3 p_\alpha^2 = 1. \quad (48)$$

Apart from the critical points shown in the table above, we have found sets of infinite points in the particular cases $\gamma = 0, 1, 2$, which at the same time constitute bifurcations and will be discussed later. From the eigenvalues we get the dynamical character of the critical points, which is shown in the table below

Model	Dynamical character		
	$0 < \gamma < 1$	$\gamma = 1$	$\gamma > 1$
F_\pm	saddle	saddle	saddle
dS_+	attractor	attractor	attractor
dS_-	repeller	repeller	repeller
K_+	repeller	repeller	saddle
K_-	attractor	attractor	saddle
m_+	saddle	repeller	repeller



FIG. 6. State space for Bianchi I models with $\gamma = 0$.

m- saddle attractor attractor

Now let us analyze the state space for these models. It can be represented by the same drawings used for the ${}^3R \leq 0$ sector of the FLRW evolution, the only thing we need to change is the axis corresponding to the variable Ω_k . For the Bianchi I models instead of Ω_k we have to consider Ω_σ , and instead of the critical points M_\pm (Milne), we have to consider K_\pm (Kasner). Then, taking into account this correspondence, let us examine the structure of the state space for Bianchi I models for the different values of γ . For the sake of simplicity, we will do it only for expanding models, that is $\epsilon = 1$. The case $\epsilon = -1$ can be obtaining by a simple time reversal.

In the case $\gamma = 0$, all the points given by $\Omega^* = (\Omega_\rho^*, \Omega_\Lambda^*, 0, \Omega_\lambda^*)$ such that $\Omega_\rho^* + \Omega_\Lambda^* + \Omega_\lambda^* = 1$, are critical points corresponding to the de Sitter model as given in expressions (33,34) for $k = 0$. The dynamics can be shown in an one-dimensional state space (see Figure 6). De Sitter is the attractor and Kasner the repeller, therefore, any initial anisotropy is diluted out in the evolution.

The situation in the case $\gamma \in (0, 1)$ is more complicated. The state space is represented in Figure II(b). As we can see, F_+ is the attractor in the case without cosmological constant. When a cosmological constant is present, the de Sitter is the general attractor. In both cases, the anisotropic Kasner models are repellers of the evolution, which means that independently of the initial conditions the models isotropize.

When $\gamma = 1$, which corresponds to dust matter, a bifurcation occurs and the state space is now given in Figure II(b). The dynamical behaviour is essentially the same as in the previous case, the difference is that now we have an infinite set of critical points situated in the line $\Omega_\sigma + \Omega_\lambda = 1$. We can find these models by using the limiting procedure introduced in (31). The line element of these models is described by the Kasner metric (47),

but now only the first condition in (48) holds, i.e.,

$$\sum_{\alpha=1}^3 p_{\alpha} = 1, \quad (49)$$

where the parameters p_{α} depend on $\kappa_{(5)}$, and in the limit $\rho \rightarrow 0$, where the influence of the extra dimension disappears, we recover the vacuum Kasner models.

In the interval $\gamma \in (1, 2)$ the only change with respect to the situation in the case $\gamma \in (0, 1)$ is that now the BDL model (m_+) is a repeller point and K_{\pm} are saddle points. The state space is represented in Figure II(b). In $\gamma = 2$, which corresponds to a stiff matter equation of state ($p = \rho$), there is another bifurcation. We have a line of general relativistic critical points, as shown by Figure II(b). The models are described, as in the previous case ($\gamma = 1$), by a Kasner metric (47) where the parameters only satisfy (49) and the energy density is given by

$$\kappa^2 \rho = \frac{1}{2t^2} \left(1 - \sum_{\alpha=1}^3 p_{\alpha}^2 \right). \quad (50)$$

They are saddle points. The dynamics of the rest of the state space is as in the case $\gamma \in (1, 2)$. Finally, for $\gamma > 2$ (which does not satisfy the causality condition), the state space would be given by Figure II(b).

To sum up, we have seen that expanding models isotropize as it happens in general relativity, although now we can have intermediate stages in which the anisotropy can grow [see Figure II(b) for $\gamma \in (1, 2)$]. The situation near the Big Bang is more interesting. In the brane-world scenario anisotropy dominates only for $\gamma < 1$, whereas in general relativity it dominates for $\gamma < 2$, therefore in the physically relevant interval $\gamma \in (1, 2)$ the prediction is completely different (for the context of inflation see [30]), in the brane-world scenario the singularity is isotropic.

B. Bianchi V perfect-fluid cosmologies

In the Bianchi V cosmological models the hypersurfaces of homogeneity, which we have assumed to be orthogonal to the fluid velocity, are negatively curved. In fact, we can pick up a triad $\{e_{\alpha}\}$ Fermi-Walker propagated along \mathbf{u} and such that the spatial commutation functions satisfy

$$a_1 \neq 0, \quad a_2 = a_3 = 0, \quad \text{and} \quad n_{\alpha\beta} = 0. \quad (51)$$

Then, the spatial scalar curvature (36) is given by

$${}^3R = -6a_1^2 < 0.$$

We can introduce the quantity Ω_k as defined in (12), but now it looks as follows

$$\Omega_k \equiv -\frac{{}^3R}{6H^2} = \frac{a_1^2}{H^2} > 0.$$

Therefore, using also the variables Ω_{ρ} , Ω_{Λ} , Ω_{λ} [equations (12,13)], and Ω_{σ} (40), the Friedmann equation (35) becomes

$$\Omega_{\rho} + \Omega_k + \Omega_{\Lambda} + \Omega_{\sigma} + \Omega_{\lambda} = 1.$$

This equation implies that we can construct a compact state space from the variables $\mathbf{\Omega} = (\Omega_{\rho}, \Omega_k, \Omega_{\Lambda}, \Omega_{\sigma}, \Omega_{\lambda})$, which are all positive, and as usual, restricted to the interval $[0, 1]$.

Before looking at the dynamical system for $\mathbf{\Omega}$ let us consider the constraint (37). In this case it is not automatically satisfied, but it imposes, by virtue of (51), the following condition

$$\sigma_{1\alpha} = 0, \quad (52)$$

which supposes a restriction on the general metric of the Bianchi V models, whose line element can be written in the following way

$$ds^2 = -dt^2 + A^2(t)dx^2 + e^{2x} [B^2(t)dy^2 + C^2(t)dz^2].$$

The restriction imposed by (52) is then

$$A^2 = BC. \quad (53)$$

To find the dynamical system we need the evolution equations for a_1 and σ^2 . The equation for a_1 is [29]

$$\dot{a}_1 = -Ha_1,$$

and the equation for σ^2 is (41). Then, the equations for $\mathbf{\Omega}$ are

$$\Omega'_{\rho} = \epsilon[2(1+q) - 3\gamma]\Omega_{\rho}, \quad (54)$$

$$\Omega'_k = 2\epsilon q \Omega_k, \quad (55)$$

$$\Omega'_{\Lambda} = 2\epsilon(1+q)\Omega_{\Lambda}, \quad (56)$$

$$\Omega'_{\sigma} = 2\epsilon(q-2)\Omega_{\sigma}, \quad (57)$$

$$\Omega'_{\lambda} = 2\epsilon[1+q-3\gamma]\Omega_{\lambda}, \quad (58)$$

where the deceleration parameter is also given by the expression (46). The critical points of the dynamical system (54-58) as well as their coordinates and eigenvalues are given in the following table [28]

Model	Coordinates	Eigenvalues
F_{ϵ}	$(1, 0, 0, 0, 0)$	$\epsilon(3\gamma - 2, 3\gamma - 2, 3\gamma, 3(\gamma - 2), -3\gamma)$
M_{ϵ}	$(0, 1, 0, 0, 0)$	$-\epsilon(3\gamma - 2, 0, -2, 4, 2(3\gamma - 1))$
dS_{ϵ}	$(0, 0, 1, 0, 0)$	$-\epsilon(3\gamma, 2, 2, 6, 6\gamma)$
K_{ϵ}^*	$(0, 0, 0, 1, 0)$	$\epsilon(-3(\gamma - 2), 4, 6, 4, -6(\gamma - 1))$
m_{ϵ}	$(0, 0, 0, 1, 0)$	$2\epsilon(\frac{3\gamma}{2}, 3\gamma - 1, 3\gamma, 3(\gamma - 1), 3\gamma - 1)$

That is, we recover the equilibrium points we had in the case of FLRW models with ${}^3R < 0$ plus the models denoted by K^* , which corresponds to Kasner models. However, we must take into account the restriction (53),

[height=0.20in,width=2.4in]bianchi-v-bis 0_g amma $_w$ l.eps
FIG. 7. State space for Bianchi V models with $\gamma = 0$.

which implies that the critical points K^* only represent Kasner models for which the parameters p_α are given by

$$p_1 = \frac{1}{3}, \quad p_2 = \frac{1 + \sqrt{3}}{3}, \quad p_3 = \frac{1 - \sqrt{3}}{3}. \quad (59)$$

On the other hand, we have now sets of infinite points for $\gamma = 0, \frac{1}{3}, \frac{2}{3}, 1, 2$, which also are bifurcation values of the parameter γ . We have more bifurcations than in the Bianchi I case, so we need more state space diagrams to represent the dynamics. To do that we need to extract, from the previous table, the dynamical character of the equilibrium points, which is shown in the next table

Model	Dynamical character		
	$0 < \gamma < 1$	$\gamma = 1$	$\gamma > 1$
F_\pm	saddle	saddle	saddle
M_\pm	saddle	saddle	saddle
dS_+	attractor	attractor	attractor
dS_-	repeller	repeller	repeller
K_+	repeller	repeller	saddle
K_-	attractor	attractor	saddle
m_+	saddle	repeller	repeller
m_-	saddle	attractor	attractor

Let us now analyze the state space diagrams shown in Figures III B-III B. First of all we notice that to get diagrams similar to those of the FLRW and Bianchi I models we would need four-dimensional diagrams. However, this is not necessary since, as it happens in the case of the FLRW and Bianchi I models, the qualitative dynamics follows from the trajectories of two-dimensional invariant submanifolds. Hence, we have drawn two-dimensional state space diagrams in which all the dynamical information is present. We have drawn only the trajectories joining critical points and the direction of the dynamical flow. The interior trajectories can be derived from them and by comparison with the state space diagrams for the FLRW and Bianchi I models.

We start with the case $\gamma = 0$, in which the equation (10) implies that the energy density is constant. The state space is again very simple (see Figure III B), the de Sitter model, with a modified cosmological constant (34), is the general attractor and the Kasner and Milne spacetimes are repellers.

Now, let us consider the case $\gamma \in (0, \frac{1}{3})$, which is represented in Figure III B. Again, for the sake of brevity we will consider only the expanding case, i.e., $\epsilon = 1$ (the case $\epsilon = -1$ is obtained by time reversal, i.e. by reversing the arrow in the state-space diagrams). The attractor for $\Lambda = 0$ is the flat Friedmann model, whereas for $\Lambda \neq 0$ is de Sitter. The other critical points are saddle points. As in the Bianchi I case, these models isotropize (evolving either to F_+ or dS_+), with the exception of extreme

[height=3in,width=3in]bianchi-v $_0 - 0.3_g$ amma $_w$ l.eps
FIG. 8. State space for Bianchi V models with $\gamma \in (0, \frac{1}{3})$. Changing the line joining m_+ and M_+ by a line in which all the points are critical we get the state space for $\gamma = \frac{1}{3}$. Reversing the arrow in that line we have the state space for $\gamma \in (\frac{1}{3}, \frac{2}{3})$.

[height=3in,width=3in]bianchi-v $_0.6_g$ amma $_w$ l.eps
FIG. 9. State space for Bianchi V models with $\gamma = \frac{2}{3}$.

situations of zero measure, representing models evolving towards the BDL solution. For $\gamma = \frac{1}{3}$ we have the first bifurcation due to the appearance of a line with an infinite number of critical points located at $\Omega^* = (0, \Omega_k^*, 0, 0, \Omega_\lambda^*)$ with $\Omega_k^* + \Omega_\lambda^* = 1$ (see Figure III B). They are the models discussed in the FLRW case (see Figure II). In the case $\gamma \in (\frac{1}{3}, \frac{2}{3})$ the only change with respect to the case $\gamma \in (0, \frac{1}{3})$ is that now models in the line joining m_+ and M_+ evolve to M_+ instead of m_+ .

For $\gamma = 2/3$ we have another bifurcation (see Figure III B) with a set of equilibrium points located at $(\Omega_\rho^*, \Omega_k^*, 0, 0, 0)$, with $\Omega_\rho^* + \Omega_k^* = 1$. They are FLRW models that correspond to the critical points in Figure II(b).

In the interval $\gamma \in (\frac{2}{3}, 1)$ (Figure III B) we have that the flat FLRW model is a saddle point, even if we restrict ourselves to the plane $\Omega_\Lambda = 0$. For $\gamma = 1$ we have a bifurcation characterized by the appearance of a set of anisotropic critical points which are not general relativistic in nature since their coordinates are $(0, 0, 0, \Omega_\sigma^*, \Omega_\lambda^*)$ with $\Omega_\sigma^* + \Omega_\lambda^* = 1$. They are among the critical points discussed in the case $\gamma = 1$ of Bianchi I. The metric is given by the line element (47), but now the exponents p_α satisfy

$$p_1 = \frac{1}{3} \quad \text{and} \quad p_2 + p_3 = \frac{2}{3}. \quad (60)$$

In the situation $\gamma \in (1, 2)$ (Figure III B), the only change with respect to the case $\gamma \in (\frac{2}{3}, 1)$ is that the points K_+ are saddle instead of repellers. This means that if we initially start with $\Omega_\omega^* + \Omega_\lambda^* = 1$, the models will evolve towards K_+ instead of m_+ .

Finally, for $\gamma = 2$ we have again an infinite number of critical points (see Figure III B). They are spatially-flat models and therefore, following the discussion of the Bianchi I models, they are described by a Kasner met-

[height=3in,width=3in]bianchi-v $_0.6 - 1_g$ amma $_w$ l.eps
FIG. 10. State space for Bianchi V models with $\gamma \in (\frac{2}{3}, 1)$. Changing the lines joining m_+ and K_+ by lines in which all the points are critical we get the state space for $\gamma = 1$. Reversing the arrow in those lines we have the state space for $\gamma \in (1, 2)$.

ric (47) with exponents given by (60) and energy density by (50).

To summarize, we can say that the dynamics of the Bianchi V models encompasses the features of the state spaces of the FLRW models with ${}^3R \leq 0$ and the Bianchi I models.

IV. REMARKS AND CONCLUSIONS

In this paper we have studied systematically the dynamics of homogeneous cosmological models (the FLRW and Bianchi I and V models) in a generalized version of the scenario proposed by Randall and Sundrum [1], stressing the main differences with respect to the general-relativistic case. In the case of the FLRW cosmological models, the state space presents a new equilibrium point, namely, the BDL model (m_{\pm}) [8]. It dominates the dynamics at high energies, where the extra-dimension effects become dominant. For this reason, we expect them to be a generic feature of the state space of more general cosmological models in the brane-world scenario, as it occurs in the Bianchi models analyzed here. In the FLRW case the critical points m_{\pm} describe the new dynamics near the Big Bang and also near the Big Crunch for recollapsing models.

Another new feature is the existence of new bifurcations as we change the equation of state, the parameter γ . In the case of FLRW models there is one new bifurcation for $\gamma = \frac{1}{3}$, characterized by the appearance of an infinite number of non-general-relativistic critical points. Among them we find a static model whose line-element is that of the Einstein universe. This contrasts with general relativity, where it appears for $\gamma = \frac{2}{3}$. The consequence is that in the brane-world scenario recollapsing models appear for $\gamma > \frac{1}{3}$ instead of $\gamma > \frac{2}{3}$ as in general relativity. In the case of Bianchi I models we have found a new bifurcation for $\gamma = 1$, and in the case of Bianchi V models for $\gamma = \frac{1}{3}, 1$.

On the other hand, Bianchi models allow us to study anisotropy. We have seen that expanding Bianchi I and V models always isotropize, as it happens in general relativity, although now we can have intermediate stages in which the anisotropy grows. This is an expected result since the energy density decreases and hence, the effect of the extra dimension becomes less and less important. The situation changes drastically when we look backwards. Near the Big Bang the anisotropy only dominates for $\gamma \leq 1$, whereas in general relativity it dominates for $\gamma < 2$, which includes all the physically interesting cases.

Just to finish we would like to mention some current

and future work in the line of the present one. First we recall that in this paper we have considered brane-world scenarios in which the bulk satisfies the condition (7). Then, it would be interesting to look at the effect of having a contribution from the bulk curvature, or in other words, a contribution from the bulk Weyl tensor piece $E_{ab}^{(5)}$. This is currently under investigation [20]. On the other hand, taking into account that string theory is formulated in spacetimes with more than one extra dimension (brane world of codimension greater than one) it would be interesting to study how the introduction of more extra dimensions changes the results presented here. In this sense, a good starting point would be to consider scenarios like those introduced in [31].

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